

On the New Theory of Integration.

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§ 1. In a paper published in the 'Proceedings of the London Mathematical Society,'* addressed to persons already acquainted with Lebesgue integration, I endeavoured to show that the method of monotone sequences enabled us to recognise intuitively the extensibility to Lebesgue integration of results known to be true for Riemann integrals. For this purpose I naturally employed known results in the Theory of Sets of Points; and, of course, also pre-supposed the proofs of the classical theorems whose generalisation was in question.

In the present communication I propose to indicate briefly how the method of monotone sequences enables us to prove, at one and the same time, these theorems and their generalisations. For this purpose we have only to employ a slight modification of the procedure indicated in the paper cited; one which, however, avoids all reference to the Theory of Sets of Points,† and assumes no results whatever in the Theory of Integration.

A careful study of the classical treatment of the theory of the integration of a continuous function shows that it is based on two principles, which are, however, not explicitly alluded to:—

(1) The function whose integral is required is approached as limiting function by discontinuous functions, whose integrals are already known, being in fact synonymous, if there be only one independent variable, with the sum of a finite number of rectangular areas; these functions, which are constant in each of a finite number of stretches, by a natural and convenient choice of values at the points, necessarily finite in number, at which there is ambiguity, are, at our will, upper or lower semi-continuous functions.‡

(2) The mode in which the limiting function is approached is by means of monotone sequences of these functions, and it is shown that, whatever

* "On a New Method in the Theory of Integration," 'Lond. Math. Soc. Proc.,' 1910, Ser. 2, vol. 9, pp. 15–50.

† It is at the request of a distinguished mathematician of the older school that I have undertaken to explain the possibility of doing this. I hope that the novelties in the present account will be regarded as sufficiently justifying the publication of this communication.

‡ For the definition and properties of semi-continuous functions see my tract "On the Fundamental Theorems of the Differential Calculus," 'Camb. Univ. Press,' 1910, pp. 6, 7.

monotone sequence of functions of the elementary type in question be employed, the limit of their integrals is necessarily the same.

If we then turn to the work of Darboux, to whom, and not to Riemann, our theory in its most generalised form is chiefly indebted, we find that he tacitly replaces the discontinuous function of which he is desirous to define the integral by two functions, one a lower semi-continuous function not greater than the given function, and the other an upper semi-continuous function not less than the given function. To each of these functions he may be said to apply one half of the process employed in the classical theory in the case of a continuous function. The lower semi-continuous function is approached from below, and the upper semi-continuous function from above only. It follows, however, from the theory of such functions; and, in particular, from the fact that, if we divide an interval into a finite number of parts, and select a point in each part at which the function assumes its minimum value in that part—supposing it to be a lower semi-continuous function—or its maximum in that part—supposing it to be an upper semi-continuous function—and then subdivide each of the parts so obtained, and carry on the process indefinitely, the value of the function at every point is the unique limit of these minima, or maxima, respectively in its neighbourhood—it follows, I say, from this fact, that the upper and lower semi-continuous functions in question are actually the limits of the discontinuous functions approaching them. Moreover, the mode of approach is a monotone one; and, as Darboux virtually shows, the limit of the integrals of the auxiliary functions approaching from below, and the limit of the integrals of the auxiliary functions approaching from above, are each of them unique and independent of the particular auxiliary functions, of the types specified, chosen to approximate to the given function. These two limits are not, however, in general the same, and Darboux gives to them the names of *intégrale par défaut* and *intégrale par excès* respectively.

It lies in the nature of things to call these *intégrales par défaut et. par excès* the integrals of the lower and upper semi-continuous functions respectively which the generating functions have as limits. Yet, as soon as we have done this, we have already taken a step into a domain which was closed to Riemann.

If further justification for this step is needed, it is easy to give it. The extension to unbounded functions of the theory of integration in its most successful form, as developed by de la Vallée Poussin, involved monotone sequences before the idea of Lebesgue integration had been mooted. On the other hand, if for a moment we turn to the Theory of Sets of Points, we shall find that the definition we have just given of the integral of an

upper semi-continuous function, for example, corresponds precisely, in the Theory of Sets of Points, to the convention by which we attach to a *closed* set of points a content. Now, even Riemann and his disciples have, implicitly or explicitly, always admitted the logical character of this convention. The identity in question is evident, if we reflect that the function which is unity at every point of a closed set of points in the interval of integration, and zero elsewhere (*i.e.* in the complementary intervals) is an upper semi-continuous function, and that the definition we have given of integral is at once seen to be concomitant with that of the content of the closed set in question.

The matter becomes still plainer if we take the *internal* points of any non-overlapping set of intervals, infinite in number, and not in general abutting, and suppose a function to be defined to have the value unity at each such point, and zero elsewhere. The function so obtained is lower semi-continuous, and its integral, as we have defined it, is nothing more nor less than the sum of the series, necessarily convergent, of the lengths of these intervals.

The moment, however, that our definitions of the integrals of upper and lower semi-continuous functions have been accepted, the whole theory develops itself without a hitch. In the first place we naturally formulate the following principle, which is perfectly general, and applies equally to bounded and unbounded functions:—

(I) *A function is said to have an integral if it can be expressed as the limit (finite or infinite with determinate sign) of a monotone succession of functions, belonging to a class of functions whose integrals have already been defined, provided only that the limit of the integrals of the functions of every such succession is the same, and this limit is then called the integral of the given function.*

It is convenient to add the gloss that the integral is not regarded as existing unless it is finite.

Denoting for brevity upper and lower semi-continuous functions by the letters u and l , we are thus led to examine the nature of the functions generated as limits of monotone sequences of l -functions and u -functions. A descending sequence of u 's gives us an u , an ascending sequence of l 's gives us an l , but an ascending sequence of u 's gives us a new function in general, which we call a lu , and a descending sequence of l 's gives us a new function which we call an ul . A repetition of this process leads, on the one hand, to a re-appearance of the lu - and ul -functions, and on the other to two new types of functions, which we may call lul - and ulu -functions. It is clear that the process may be extended indefinitely, in such a manner, moreover, as to obtain functions for which a modified system of nomenclature is essential.

Confining our attention in the first instance to bounded functions, we have

to show that all bounded functions of the classes of types obtained in the manner explained possess integrals, in accordance with our principle (I). A theorem remarkable in itself, though intuitive in the light of recent theory, enables us to prove this once and for all, and at the same time gives us a different definition of integration, different in form, though equivalent in essence. The theorem follows so simply from the considerations I have exposed, that I propose to state and prove it here.

§ 2. We shall suppose that the definition of the integrals of *ul*- and *lu*-functions as the limit of the integrals of descending successions of *l*-functions, and of ascending successions of *u*-functions respectively, has been shown to be in accordance with principle (I), and that this proof has been further supplemented by a proof that, if a function is both an *ul* and a *lu*, its integral defined by both of these two distinct processes is the same. We shall also assume that term-by-term integration of monotone sequences involving functions which are *lu*- or *ul*-functions has been shown to be allowable. The proof then turns on two lemmas:—

Lemma 1.—*Given a bounded lu, we can always find an ul, nowhere less than the lu, and having the same integral, and given a bounded ul, we can always find a lu, nowhere greater than the ul, and having the same integral.*

It is evident that the one half of the statement turns into the alternative half, if we change the signs of the functions.

Let $f_1(x) \leq f_2(x) \leq \dots$ be a monotone ascending succession of *u*-functions, whose limit is the given *lu*-function $f(x)$.

Since $f_n(x)$ is an *u*-function, we may regard it as the limit of a monotone descending sequence of the elementary *l*-functions, and its integral as the limit of their integrals. We may, therefore, take an *l*-function $b_n(x) \geq f_n(x)$, where

$$\int b_n(x) dx \leq \int f_n(x) dx + 2^{-n-1}e.$$

If the succession $b_1(x), b_2(x), \dots$ is not monotone ascending, we make it so, as follows. Wherever $b_1(x) > b_2(x)$, we replace the value of $b_2(x)$ by $b_1(x)$. Let us denote the modified function by $c_2(x)$. Then $c_2(x)$ will still be $\geq f_2(x)$, also it remains an *l*-function. Moreover

$$c_2(x) - f_2(x) \leq [b_2(x) - f_2(x)] + [b_1(x) - f_1(x)],$$

whence, since both sides of this inequality represent *l*-functions,

$$\int [c_2(x) - f_2(x)] dx \leq e(2^{-2} + 2^{-3}) < \frac{1}{2}e.$$

Similarly we modify b_3, b_4, \dots . We thus get a monotone ascending sequence of *l*-functions $c_2(x), c_3(x), \dots, c_n(x), \dots$ such that

$$\begin{aligned} c_n(x) &\geq f_n(x), \\ \int [c_n(x) - f_n(x)] &\leq \frac{1}{2}e. \end{aligned} \tag{1}$$

Since the functions $c_n(x)$ and $f_n(x)$ are l -functions, this gives

$$\int f_n(x) dx \leq \int c_n(x) dx \leq \frac{1}{2}e + \int f_n(x) dx. \quad (2)$$

Hence, if $g_1(x)$ is the l -function which is the limit of the c_n -succession, we get, by the definition of the integral of the lu -function $f(x)$, from (2),

$$\int f(x) dx \leq \int g_1(x) dx \leq \frac{1}{2}e + \int f(x) dx,$$

while, from (1)

$$g_1(x) \geq f(x).$$

Now let us again perform the same construction, taking $\frac{1}{4}e$, instead of $\frac{1}{2}e$, and choosing each of the l -functions which we employ to be $\leq g_1(x)$. We thus obtain an l -function $g_2(x) \geq f(x)$ and $\leq g_1(x)$, and such that

$$\int f(x) dx \leq \int g_2(x) dx \leq \frac{1}{4}e + \int f(x) dx.$$

Continuing thus we obtain a monotone descending sequence of l -functions,

$$g_1(x) \geq g_2(x) \geq \dots \geq g_n(x) > \dots,$$

each $\geq f(x)$, and such that, for each value of n ,

$$\int f(x) dx \leq \int g_n(x) dx \leq 2^{-n}e + \int f(x) dx.$$

If $g(x)$ be the limiting function of the g_n -sequence, then $g(x)$ is $\geq f(x)$ and is, by definition, an ul -function, and its integral, being the limit of $\int g_n(x) dx$ is, as is evident from the last inequality, equal to that of $f(x)$. This proves the theorem.

Lemma 2.—*Given a bounded lul , we can always find an ul , nowhere less than the lul , and having the same integral; and given a bounded ulu , we can always find a lu , nowhere greater than the ulu , and having the same integral.*

The argument is precisely the same as the proof of Lemma 1, except that the functions $b_n(x)$ are general l -functions, instead of elementary l -functions.

§ 3. We can now at once prove the theorem which I have in mind.

Theorem.—*Given any bounded function, formed by any monotone process, such as those here described, we can find an ul -function not less than it, and a lu -function not greater than it, which have the same integral.*

Suppose, for definiteness, that the monotone succession defining the function $f(x)$ is an ascending one, say,

$$f_1(x) \leq f_2(x) \leq \dots$$

Then, without loss of generality, we may suppose that the theorem has been proved to be true for each function $f_n(x)$ of the sequence.

Let us take a lu -function, $g'_n(x) \leq f_n(x)$, and having the same integral. Doing this for each integer n , we get a new succession, which, if not already monotone increasing, we modify as follows:—At any point where $g'_1(x) > g'_2(x)$, let us increase the value of the latter to that of the former. Denoting the modified function by $g_2(x)$, it is still a lu -function; and, since

$$g'_2(x) \leq g_2(x) \leq f_2(x),$$

it has the same integral as before, equal to that of $f_2(x)$. We then proceed to similarly modify g'_3 , and so on. Thus we have a monotone ascending sequence of lu -functions $g_n(x)$, whose integrals are equal to those of the functions $f_n(x)$.

The limiting function $g(x)$ of this succession is an lu , that is a lu -function, and is, of course, like every $g_n(x)$, not greater than $f(x)$. Also its integral may be obtained by term-by-term integration of the g_n -sequence; and is, accordingly, the limit of $\int f_n(x) dx$, that is $\int f(x) dx$. Thus we have found such a lu -function as was required.

Again, let us take an ul -function $h'_n(x) \geq f_n(x)$, and having the same integral. Doing this for each integer n we get a new succession, which, if not already monotone increasing, we proceed to modify as follows:—Let $h_1(x)$ be the function whose value at each point is the lower bound of $h'_1(x)$, $h'_2(x)$, ..., $h'_n(x)$, ... This function is easily seen to be an ul -function, since it is the limit of a monotone descending sequence of ul -functions, got by taking the lower bound of a finite number of the functions $h'_n(x)$. Similarly, let $h_2(x)$ be the function whose value at any point is the lower bound of $h'_2(x)$, $h'_3(x)$, ..., $h'_n(x)$, ..., and so on. We thus get a monotone ascending sequence of ul -functions $h_n(x)$, such that

$$f_n(x) \leq h_n(x) \leq h'_n(x),$$

so that the integrals of these three functions are the same. The limiting function $h(x)$ is a lul -function, whose integral is, by definition, the limit of that of $h_n(x)$, that is of $f_n(x)$, and is therefore the same as that of $f(x)$.

§ 4. We are thus able to show immediately that all the functions of the types we have introduced possess integrals in accordance with the principles we have laid down.

This is, however, not all that the theorem just proved enables us to do. It gives us an entirely new definition of the concept of integration, which includes what we may, for convenience, call that of Darboux as a particular case. Our new definition is as follows:—

Form the integrals of all upper semi-continuous functions less than the given function, and take the upper bound of these integrals; form the integrals of all the lower semi-continuous functions greater than the given function, and take the lower bound of these integrals; then, if the upper bound of the former and the lower bound of the latter agree, the function is said to possess an integral, and the value of the integral is the common value of that upper bound and that lower bound.

The superiority of this definition over that of Darboux (or that of Riemann) consists in the fact that all bounded functions which may, in a sense easily

understood, be said to be expressible mathematically, possess an integral in accordance with the new definition.

Darboux's definition fails, in fact, because it is in general impossible to find an l -function and an u -function having the given function between them, and possessing the same integral.

The new definition succeeds because it is always possible, in all the circumstances which can arise in mathematical investigations, to find a lu -function and an ul -function between which the function to be considered lies (provided only it is bounded) having the same integral.

§ 5. When we come to consider unbounded functions no fresh difficulty arises in the application of our original principle, provided always we consider separately the modulus of the function, and the excess of the modulus over the function, or, which comes to the same thing, the two positive functions f_1 and f_2 , whose difference is f and whose sum is the modulus of f . As in the other theories non-absolutely convergent integrals require separate treatment. This treatment may be the same as in the older theory, except that a limitation is removed, corresponding to the restriction implied in the Riemann definition.

In discussing unbounded functions, we may, therefore, in considering the new theory, confine our attention to positive functions, and it is clear, from what has gone before, that this is the case also with bounded functions. In other words, in the proof of all our theorems in the new theory of integration we are at liberty to suppose that the functions with which we are concerned are positive, and we need not restrict them to be bounded. In the case of an unbounded positive function, we can no longer enclose it between an ul and a lu , we can, however, always enclose it between an unbounded lul and an unbounded lu . If we include infinity ($+\infty$) as a possible value of the integral, we can then prove that all unbounded positive functions which can present themselves in mathematical reasoning necessarily possess an integral in accordance with both our definitions.

§ 6. I have now completed my sketch of the theory, regarded from the point of view of monotone sequences. If the exposition may seem somewhat long, two things are to be observed:—One is that all reference to the Theory of Sets of Points, with the various difficulties which it presents to many students, is avoided in it: and the second is that, if the development of the theory is tedious, it is not so with the applications of it. It remains to show that this is the case. For this it will be sufficient to consider two examples of different types. I take first the theorem that change of order of integration is always allowable in the case of an unbounded positive function.

As passage by monotone sequences is always permissible when we are dealing with positive quantities, it is at once evident that the theorem is true

always, if it is true for the simple bounded upper or lower semi-continuous functions with which our chain of monotone sequences began. But these functions are constant in rectangles, and their double integrals consist of the sum of the volumes of a number of rectangular parallelopipeds. Hence, in the case of these simple functions, the repeated integrals are both equal to the double integral. Hence, they must be equal in the case of any bounded, or positive unbounded function of two variables whatever.

As another example I take Schwarz's inequality, namely, that

$$(\int uv \, dx)^2 \leq \int u^2 \, dx \int v^2 \, dx.$$

It is clearly sufficient to prove it for the simple functions.

Let a_1, a_2, \dots, a_n be the values assumed by one of these simple functions u , corresponding to a division of the interval of integration into n equal parts, each of length h ; b_1, b_2, \dots, b_n the corresponding values of v . Then

$$\int uv \, dx = (a_1 b_1 + a_2 b_2 + \dots + a_n b_n) h,$$

$$\int u^2 \, dx = (a_1^2 + a_2^2 + \dots + a_n^2) h,$$

$$\int v^2 \, dx = (b_1^2 + b_2^2 + \dots + b_n^2) h.$$

Thus the theorem is shown to be nothing more nor less than the assertion in the language of integrals of the well-known algebraic inequality,

$$(\sum a_1 b_1)^2 \leq (\sum a_1^2) (\sum b_1^2).$$

§ 7. This is not an occasion to dwell on the usefulness of the new concept, which is sufficiently evident to anyone who has had an opportunity of consulting recent mathematical literature. It will suffice if I remind my readers that one of the reasons for its original introduction was that it enabled us to find the inverse differential coefficient of a bounded function in cases when the Riemann theory failed to help us.

That any bounded sequence may be integrated term by term follows, indeed, from the fact that it can be replaced in two ways by a pair of monotone sequences, the one pair consisting of a descending one followed by an ascending one, and the remaining pair by an ascending one followed by a descending one. We may, therefore, in the case supposed, integrate the sequence

$$[f(x+h) - f(x)]/h$$

term by term, since the incrementary ratio and the differential coefficient have the same upper and lower bounds.

Denoting then by $F(x)$, the indefinite integral of $f(x)$, and integrating between the limits x and a , we thus get

$$\text{Lt}_{h \rightarrow 0} \left(\frac{F(x+h) - F(x)}{h} - \frac{F(a+h) - F(a)}{h} \right) = \int_a^x f'(x) \, dx,$$

whence, bearing in mind that $f(x)$, being continuous, is the differential coefficient of its integral, we get at once

$$f(x) - f(a) = \int_a^x f'(x) dx.$$

Here the sign of integration refers to integration of the generalised Lebesgue type, and it will be sufficiently evident from what precedes, that this equation is only then in general true when the sign of integration is interpreted in this sense. It may be remarked that $f'(x)$ is both a *lu* and an *ul*, and is, therefore, of a very elementary type in our scheme of functions.

On the Formation of Usually Convergent Fourier Series.

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§ 1. Series which converge except at a set of content zero, or, using the expression very commonly adopted, series which converge usually, possess many of the properties which appertain to series which converge everywhere. It becomes, therefore, of importance to devise circumstances under which we can assert the consequence that a series converges in this manner. The subject has recently received considerable attention. So far as Fourier series are concerned no result of even an approximately final character has been obtained. It may be supposed, indeed, that the results* of Jerosch and Weyl were at first so regarded,† but, if we examine them closely in the light of the Riesz-Fischer theorem, which was known previously to the results of these authors, it becomes evident that they are merely *equivalent* to the statement that the Fourier series of a function, whose square is summable, is changed into one which converges usually, if the typical coefficients a_n and b_n are divided by the sixth root of the integer n denoting their place in the series. Now it is difficult to believe that the question of the usual convergence of a Fourier series can depend on the degree of the summability of the function with which it is associated,

* 'Math. Ann.,' vols. 66 and 67.

† The result due to Fatou that a series of Fourier converges usually if na_n and nb_n converge to zero is still more special, being of course included in Jerosch's condition. For Fatou's paper, see 'Acta Mat.,' vol. 30.